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INFLUENCE OF BACKGROUND VERTICAL MOTIONS ON FREE CONVECTION OVER A THERMALLY INHOMOGENEOUS HORIZONTAL SURFACE

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In a linear approximation the problem of the influence of background vertical motions on free convection over a thermally inhomogeneous horizontal surface has been solved analytically. The dimensionless number that determines this influence and that represents the ratio of the product of the characteristic velocity of background vertical motions and of the horizontal scale of thermal inhomogeneities to the coefficient of thermal diffusivity of the medium has been found. Descending motions may effectively suppress convection (not only decrease the height of penetration of convective motions into the medium but also to decrease their amplitude substantially). Ascending motions increase the height of penetration of convective motions into the medium and, to a lesser extent, the vertical component of convective velocity.

Introduction. From experience and theory it is known that even very slow background vertical motions may noticeably influence convection, in particular, convective stability. This is important for a number of technical and geophysical applications [1–5]. Since the interaction of the processes of different spatial scales are considered, the corresponding theoretical problems are usually very complex. We considered the case where the linearized problem admits an explicit analytical solution. The influence of background vertical motions on free convection in a homogeneous incompressible (Boussinesq") medium over a thermally inhomogeneous horizontal surface is investigated.

Statement of the Problem. A semiinfinite layer of a medium $z \geq 0$ is considered (the z axis is directed vertically upward). According to the usually used approximation we assume that the density of the medium depends linearly on the perturbations of temperature T:

$$\rho = \rho_* (1 - \alpha T) .$$

The system of equations of the hydrothermodynamics in the Boussinesq approximation has the form

$$\frac{d\mathbf{v}}{dt} = -\nabla\Phi + \nu\nabla^2\mathbf{v} + g\alpha T\mathbf{e}_z , \quad \mathrm{div}\,\mathbf{v} = 0 , \quad \frac{dT}{dt} = \kappa\nabla^2 T ,$$

where $\mathbf{v}$ is the velocity vector with the components u, v, w along the x, y, and z axes, respectively; $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ is the operator of the full derivative.

We shall limit ourselves for simplicity to a two-dimensional problem with dependence only on the x and z coordinates. Stationary perturbations associated with the given temperature inhomogeneity of the horizontal surface, $z = 0$, are investigated:

$$T = T_0 \cos (kx) . \tag{1}$$

Convection against the Background of a Medium at Rest. First we consider convection (thermal circulations over a thermally inhomogeneous surface) against the background of a medium at rest. The system of the equations linearized over two-dimensional stationary perturbations has the form

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$$0 = -\frac{\partial \Phi}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \,, \tag{2}$$

$$0 = -\frac{\partial \Phi}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) + g\alpha T \,, \quad 0 = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right). \tag{3}$$

On the $z = 0$ surface, apart from Eq. (1) the non-flow and no-slip conditions $u = w = 0$ at $z = 0$ are prescribed. It is assumed that far from the surface (for $z \to \infty$) all the perturbations decay.

For the perturbations we seek stationary solutions that harmonically depend on the horizontal coordinate x:

$$u\,(x, z) = U\,(z) \sin\,(kx) \,, \quad w\,(x, z) = W\,(z) \cos\,(kx) \,, \quad T\,(x, z) = \theta\,(z) \cos\,(kx)$$

and so on. The solution of the last equation of (3) with account for the boundary-value conditions is

$$T = T_0 \exp\,(-\,kz) \cos\,(kx) \,. \tag{4}$$

At the well-known temperature field, Eqs. (2) and (3) are transformed into a closed inhomogeneous system of ordinary differential equations with constant coefficients. Having solved it, we find the perturbations [6]:

$$u = -\frac{\alpha g T_0}{8\nu k} z\,(2 - kz) \exp\,(-\,kz) \sin\,(kx) \,, \tag{5}$$

$$w = \frac{\alpha g T_0}{8\nu} z^2 \exp\,(-\,kz) \cos\,(kx) \,, \tag{6}$$

$$p' = -\frac{\rho_* \alpha g T_0}{4k} (3 - 2kz) \exp\,(-\,kz) \cos\,(kx) \,. \tag{7}$$

Perturbations penetrate into a medium up to heights of the order of the horizontal scale of perturbations $L = 2\pi/k$. It is seen that over the inhomogeneously heated surface circulation cells ("rolls") appear with ascending motions over more heated regions, with an aspect ratio of the order of unity. (We show that in the case of stable stratification of the density the vertical scales of the cells decrease.) The amplitudes of the perturbations of the velocity components u and w are of the order of $\alpha g T_0/8\nu k^2$.

Solution in the Presence of Background Vertical Motions. We will consider an analogous problem in the presence of the background vertical motion with velocity W_0 = const. The assumption on the constancy of the vertical velocity means giving up the nonflow condition on the horizontal boundary $z = 0$ for the background motion. Such situations can be realized by pumping a liquid in the vertical direction through porous horizontal boundaries. Such a model was used, in particular, in [1–4]; it is also of interest as a certain approximation to the situations with a variable background vertical velocity. The linearized system of equations in this case has the form

$$W_0 \frac{\partial u}{\partial z} = -\frac{\partial \Phi}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \,; \tag{8}$$

$$W_0 \frac{\partial w}{\partial z} = -\frac{\partial \Phi}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) + g\alpha T \,, \quad W_0 \frac{\partial T}{\partial z} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right). \tag{9}$$

We will consider the same boundary-value conditions as above and also seek solutions that harmonically depend on the horizontal coordinate x. The solution of the last equation from (9) has the form

$$T = [C_1 \exp(\sigma_1 z) + C_2 \exp(\sigma_2 z)] \cos(kx) , \tag{10}$$

the roots of the characteristic equation are

$$\sigma_{1,2} = \frac{1}{h} \pm \sqrt{\frac{1}{h^2} + k^2} , \tag{11}$$

where $h = 2\kappa / W_0$ is the length scale (positive or negative, depending on the sign of the background vertical motion W_0). Irrespective of the sign of h, the value of σ_1 (the root corresponding to the positive sign in front of the radical in (11)) is nonnegative. Therefore, in order to satisfy the condition of the decay of perturbations at high values of z, we should assume that $C_1 = 0$, so that the solution for the temperature has the form

$$T = T_0 \exp(-skz) \cos(kx) , \tag{12}$$

where

$$s = -\frac{\sigma_2}{k} = \omega + \sqrt{1 + \omega^2} ; \quad \omega = -(kh)^{-1} = -\frac{W_0}{2\kappa k} . \tag{13}$$

It is useful to keep in mind also the following relation between the dimensionless parameters ω and s: $\omega = (s^2 - 1)/2s$. Below, we will limit ourselves for simplicity to the case $\nu = \kappa$. Similarly to Eqs. (5)–(7) we find the solution

$$u = V \left[\frac{1}{s-1} (\exp(-kz) - \exp(-skz)) - skz \exp(-skz) \right] \sin(kx) , \tag{14}$$

$$w = V \left[\frac{1}{s-1} (\exp(-kz) - \exp(-skz)) - kz \exp(-skz) \right] \cos(kx) , \tag{15}$$

$$p' = \rho_* k V \nu \left[-(1 + s^2) \exp(-skz) + \frac{s+1}{s} \exp(-kz) \right] \cos(kx) , \tag{16}$$

$$V = \frac{\alpha g T_0}{2\nu k^2 \omega (1 + s^2)} .$$

Discussion of Results. As compared to Eqs. (4)–(7), the solution, apart from exp $(-kz)$, also contains the exponent exp $(-skz)$. The influence of background vertical motions is determined by the value of dimensionless parameter ω (when $\omega \to 0$, $s \to 1$). We note the dependence of ω on the wave number k — the effect of background vertical motions is enhanced with increase in the scales of convection (thermal circulations). The influence of vertical motions is noticeable even at relatively small values of $|\omega|$. This is seen, e.g., from Fig. 1, which (except for curve 2) relates to the cases of descending background motions ($\omega > 0$). Weak background subsidence noticeably weakens convection (as is seen from comparison of curves 3 and 1), whereas weak ascending motions enhance it appreciably (curve 2). In these cases, the maximum absolute values of w change substantially, but the height of this maximum changes little. Stronger background motions ($|\omega| \geq 1$) influence the latter.

We will consider in detail the case of a rather strong background subsidence ($\omega >> 1$). For this asymptotics there correspond $s \approx 2\omega >> 1$ and $V \sim \omega^{-3}$. Therefore, the amplitudes of expressions (14)–(16) decrease rapidly with an increase in ω — the convection is suppressed by the background subsidence. This is seen also from Fig. 1 (the amplitudes of the velocity components of convection u and w at $\omega = 5$ in this figure are amplified 200 and 1000 times,

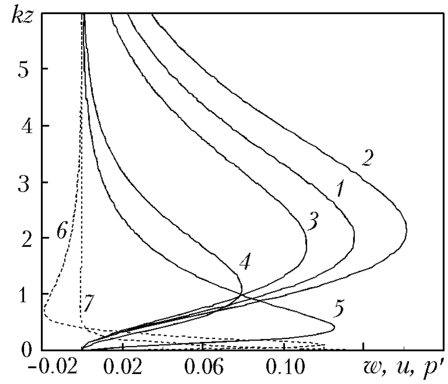

Fig. 1. Dimensionless vertical profiles of perturbations at $x = 0$ (over the heated portion of the horizontal surface) for different values of the parameter ω (intensity of background vertical motions): 1–5) the profiles of $w(z)$ normalized to V: $\omega = 0$ (1), -0.1 (2), 0.1 (3), 1 (4, the values of w are amplified five times), 5 (5, the values of w are amplified 1000 times); 6 and 7, the profiles of $-u(z)$ (normalized to $V/200$) and p' (normalized to $4\rho_* kV\nu/3$), respectively at $\omega = 5$.

respectively). Expressions (14)–(16), apart from $\exp(-kz)$, also contain the rapidly decreasing exponent $\exp(-skz)$, but the relative weight of these exponents in the fields of the perturbations of pressure, temperature, and velocity components is different. According to Eq. (12), in the temperature field there is only $\exp(-skz)$. In the pressure field there is also the relatively slowly decreasing exponent $\exp(-kz)$ but with a substantially smaller weight. Therefore, in Fig. 1 the region of reduced pressure over the heated portion of the surface looks "compressed" to the surface $z = 0$ by background subsidence (curve 7). In the field of the horizontal velocity u the weight of these exponents also differs strongly but less than in the field of p'. Finally, in the field of w the coefficients at the exponents mentioned are of comparable magnitude — the vertical motions go far beyond the thin layer (of thickness of the order of $(sk)^{-1}$) in which the perturbations of other quantities are enclosed (curve 5). The maximum of the profile of w is located in this case at a height of the order of $(sk)^{-1}$, i.e., as the background descending motions increase, this maximum decreases, although not to a such extent as does the convection amplitude.

Now, we shall consider the case of background ascending motions ($\omega < 0$, Fig. 2). At higher (in absolute magnitude) values of ω we obtain the asymptotics $s \approx 1/2|\omega| \ll 1$, $V \sim -1/|\omega|$. In expressions (14)–(16), apart from $\exp(-kz)$, there is a slowly decreasing exponent $\exp(-skz)$, so that convection against the background of such ascending motions penetrates into the medium higher (up to the levels of the order of $(sk)^{-1}$). The relationship between the absolute values of the coefficients at the mentioned exponents in the fields of the perturbations of pressure, temperature, and velocity components are different in this case too. In the expression for the temperature only a slowly decreasing exponent $\exp(-skz)$ is present. It prevails also in the expression for w. At $z \gg k^{-1}$ in the square brackets in Eq. (15) the latter term is the main one, so that

$$w \approx -Vkz\exp(-skz)\cos(kx) \approx \frac{\alpha g T_0}{\nu k^2} skz\exp(-skz)\,.$$

As ω increases (s decreases), the position of the maximum of the latter function increases as $(sk)^{-1}$, but the maximum value of the function does not change in this case. In the square brackets in expression (14) for u the maximum values of all the terms are of the order of unity, whereas the factor V before the brackets decreases in absolute magnitude with $|\omega|$ proportionally to $-1/|\omega|$. Thus, in contrast to other fields, the horizontal velocity of convective motions decreases not only in the case of the background subsidence of the medium, but also with the strengthening of the background ascending motions. As for the perturbations of pressure, in expression (16) at small values of s the exponent $\exp(-kz)$ enters with a high weight. Therefore, in contrast to other fields the perturbation of pressure is mainly concentrated in a relatively thin layer with vertical dimensions of the order of k^{-1}. The amplitude of this perturbation

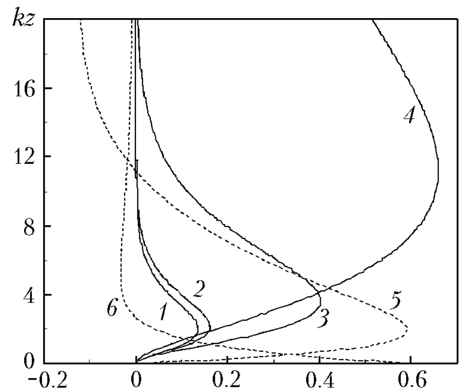

Fig. 2. Dimensionless vertical profiles of perturbations at $x = 0$ in the case of ascending background motions: 1–4) the profiles of $w(z)$ normalized to V: ω = 0 (1), –0.1 (2), −1 (3), −5 (4); 5 and 6, the profiles of $-u(z)$ (normalized to $V/5$) and of $p'(z)$ (normalized to $3\rho_* kV\nu/10$) at $\omega = -5$.

tends to a constant with an increase in $|\omega|$. It is seen that in the limit considered the system is far from the hydrostatic equilibrium — the perturbation of the temperature (density) and of pressure are distributed in layers of substantially different thicknesses.

Let, for example, $|W_0| = 10^{-2}$ m/sec (the background vertical velocity typical of large-scale atmospheric vortices — cyclones and anticyclones), $\kappa = \nu = 1$ m^2/sec (the characteristic value of the effective coefficient of turbulent exchange in the atmosphere), and $k = 3 \cdot 10^{-3}$ m^{-1} (the value corresponding to the characteristic horizontal scale of some types of convective clouds). In this case, $|\omega| \approx 1.5$. From this it is seen that very weak background vertical motions may also noticeably influence the convection of the horizontal scale considered. When these are descending motions (anticyclone), they efficiently suppress convection of large scales; when these are ascending motions, they efficiently support and enhance a large-scale convection.

Conclusions. The model considered allows one in a linear approximation to obtain a simple analytical solution for convection (thermal circulations) over an inhomogeneously heated horizontal surface and describe the influence exerted on this convection by the background vertical motions. The dimensionless number ω that determines this influence has been found. An analysis shows that the influence of the ascending and descending background motions on the convection considered is unsymmetrical. The descending motions may efficiently suppress convection (not only decrease the height of penetration of convective motions into the medium but also reduce greatly their amplitude). Ascending motions increase the height of penetration of perturbations into the medium, but to a lesser extent they influence the increase of the vertical component of the convective velocity. Therefore, the derivative $\partial w/\partial z$ decreases in absolute magnitude and (because of the continuity) the horizontal velocity component of convective motions decreases. Thus, the latter velocity decreases under the influence of background vertical motions of any sign.

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NOTATION

$C_{1,2}$, integration constants, K; $\mathbf{e}_z$, vertical curl; g, free fall acceleration, m/sec^2; h, length scale, m; k, wave number, m^{-1}; L, horizontal scale of perturbations, m; p', deviation of pressure from the hydrostatic one, Pa; s, dimensionless parameter; t, time, sec; T, perturbation of temperature, K; T_0, amplitude of temperature perturbations, K; u, velocity component in the direction of the x axis, m/sec; U, amplitude of horizontal velocity, m/sec; $\mathbf{v}$, velocity vector, m/sec; V, velocity scale, m/sec; w, velocity component in the direction of the z axis, m/sec; W, amplitude function of vertical velocity, m/sec; W_0, background vertical velocity, m/sec; x, y, z, horizontal and vertical coordinates, m; α, thermal coefficient of medium expansion, m^{-1}; $\Phi = p'/\rho_*$, normalized deviation of pressure from the hydrostatic one, m^2/sec^2; κ, ν, coefficients of transfer, m^2/sec; θ, amplitude of temperature perturbation, K; ρ, density of the medium, kg/m^3; ρ_*, average (reference) density of the medium, kg/m^3; $\sigma_{1,2}$, roots of the characteristic equation, m^{-1}; $\omega = -(kh)^{-1}$, dimensionless parameter.

REFERENCES

1. Y. Jaluria, *Natural Convection. Heat and Mass Transfer*, Pergamon Press, Oxford (1980).
2. R. C. J. Sommerville and T. G. Chen, Numerical simulation of convection with mean vertical motion, *J. Atmos. Sci.*, **36**, No. 5, 805–815 (1979).
3. T. B. Glukhovskaya and A. E. Ordanovich, Concerning the influence of mesoscale vertical flows on convection processes in the atmosphere, *Meteorol. Gidrol.*, No. 3, 31–36 (1984).
4. L. Kh. Ingel', Mechanism of convective instability of a binary mixture, *Zh. Éksp. Teor. Fiz.*, **128**, No. 1, 179–182 (2005).
5. A. P. Khain and L. Kh. Ingel', Numerical simulation of the interaction of a nonstationary divergent flow with convective processes in the boundary layer of the atmosphere over the ocean, *Izv. Ross. Akad. Nauk*, *Fiz. Atmos. Okeana*, **31**, No. 4, 496–506 (1995).
6. L. Kh. Ingel', Disturbances of a shear flow caused by interaction with convective rolls, *Izv. Ross. Akad. Nauk*, *Mekh. Zhidk. Gaza*, No. 3, 33–37 (2006).